A growth dichotomy for group algebras of free abelian by infinite cyclic groups (Developments of Language, Logic, Algebraic system and Computer Science)

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A growth dichotomy for group algebras of free abelian by infinite cyclic groups

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Abstract

We study growth in algebras, especially uniform exponential growth, extending historical results in the topic of growth in groups. We give a condition under which the group algebra of a free abelian by infinite cyclic group has uniform exponential growth.

1 Introduction

Given a finitely generated group G and a finite generating set $S=\{g_1^{\pm 1},\ldots,g_r^{\pm 1}\}$ of G, for each element $g\in G$, write $g=g_{i_1}^{n_1}\ldots g_{i_k}^{n_k}$. Define the *length* of g with respect to S to be the minimal nonnegative integer k for which such an expression of g is possible (the identity is considered to be an empty word, hence of length 0). The *growth function* $\gamma_{G,S}(n):\mathbb{Z}_{\geq 0}\to\mathbb{N}$ of G with respect to S is the number of elements of G of length at most n with respect to S. We write $\gamma_{G,S}(n)=\gamma_S(n)$ if G is understood. The growth being exponential or bounded by a polynomial is independent of the chosen generating set, so we speak of groups of *exponential* or *polynomial* growth respectively. A group which is neither is said to have *intermediate* growth.

Growth in groups was introduced independently by Schwarz in 1955 and Milnor in 1968 [8, 9]. In 1968 Wolf proved that a virtually nilpotent group has polynomial growth, and a virtually polycyclic group which is not virtually nilpotent has exponential growth [11]. In 1981 Gromov proved that a finitely generated group of polynomial growth is virtually nilpotent [6].

In 1968 Milnor asked whether there exists groups of intermediate growth. The question was answered affirmatively by Grigorchuk in 1983 [5]. In 1981 Gromov defined a group G to have *uniform exponential* growth if $\inf_S\left(\lim_n\gamma_S(n)^{1/n}\right)>1$. In the same year Gromov asked: if G has exponential growth, must it have uniform exponential growth [7]? The question was answered negatively by Wilson 2004 [10]. In the meantime, the question was answered positively for several classes of groups. In 2002 Alperin proved that a virtually polycyclic group has either polynomial or uniform exponential growth [1]. In 2005 Eskin, Moses, and Oh proved a linear group over a field of characteristic zero has polynomial or uniform exponential growth [4]. In 2008 Breuillard and Gelander proved that a linear group of any field has polynomial or uniform exponential growth [2].

2 Growth in algebras

Write $A = F[S]$ to denote that A is generated (as an algebra) by the set S over the field F. The growth function of A with respect to S is $\gamma_{A,S}(n) = \dim_F\left(\sum_{i=0}^{n} FS^i\right)$. Growth type is independent of the generating set, so as with groups, we define polynomial, exponential, and intermediate growth for algebras. Also as with groups, a commutative algebra has polynomial growth, and the free algebra on at least two letters has exponential growth. We say an algebra A has *uniform exponential growth* if

$$\inf_S \gamma_{A,S}(n)^{\frac{1}{n}} > 1.$$

There are examples of algebras of uniform exponential growth: Golod-Shafarevich algebras, group algebras of Golod-Shafarevich groups, and any algebra graded by $\mathbb{N}$ with exponential growth have uniform exponential growth [3]. An example of an algebra of nonuniform exponential growth is the group algebra of Wilson's group of nonuniform exponential growth over a field of characteristic 0.

We are particularly interested in the growth of group algebras, as FG has exponential, polynomial, or intermediate growth if and only if G does respectively. If FG has uniform exponential growth, so does G. It is unclear, and it is a motivating question of the author's research, whether the converse is true.

3 Main result

The main result concerns a condition under which the group algebra of a free abelian by infinite cyclic group has uniform exponential growth. The desired end result is an extension to group algebras of Alperin's result that a polycyclic group has either polynomial or uniform exponential growth. The free abelian by infinite cyclic group is a building block of the polycyclic group, so it makes sense to begin by attempting to establish that a group algebra of a free abelian by infinite cyclic group is of polynomial or uniform exponential growth.

Henceforth, let $\Gamma = G \rtimes_\sigma \mathbb{Z}$ be a free abelian by infinite cyclic group. Let F be a field. Since the action of σ on G can be described by a matrix, we can refer to the eigenvalues of σ.

By a result of Alperin [1], combined with the fact that $F\Gamma$ has polynomial growth if and only if Γ does, we have the result

Theorem 1 *If all eigenvalues of σ have norm 1, then the group algebra $F\Gamma$ has polynomial growth.*

The following simple result is often useful:

Lemma 2 *Uniform exponential growth lifts from homomorphic images for groups and algebras [3].*

The main result is:

Theorem 3 *If σ has a real eigenvalue λ with $|\lambda| > 1$, then $F\Gamma$ has uniform exponential growth.*

The yet unsolved case to establish the growth dichotomy (polynomial or uniform exponential) for free abelian by infinite cyclic groups is the case when all eigenvalues are complex but not all eigenvalues have norm 1.

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